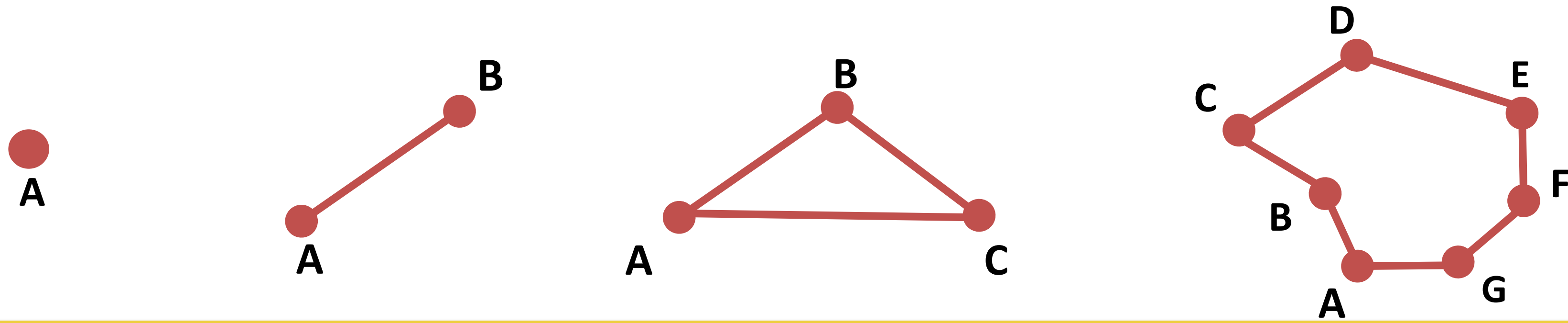
Fundamentals of Computational Geometry

CSCI 587: Lecture 2

01/15/2025

What is Computational Geometry?

- Design, Analysis and Implementation of *efficient algorithms* for solving geometric problems, e.g., problems involving points, lines, segments, triangles, polygons



Many Applications

Robotics

Geographic
Information Systems

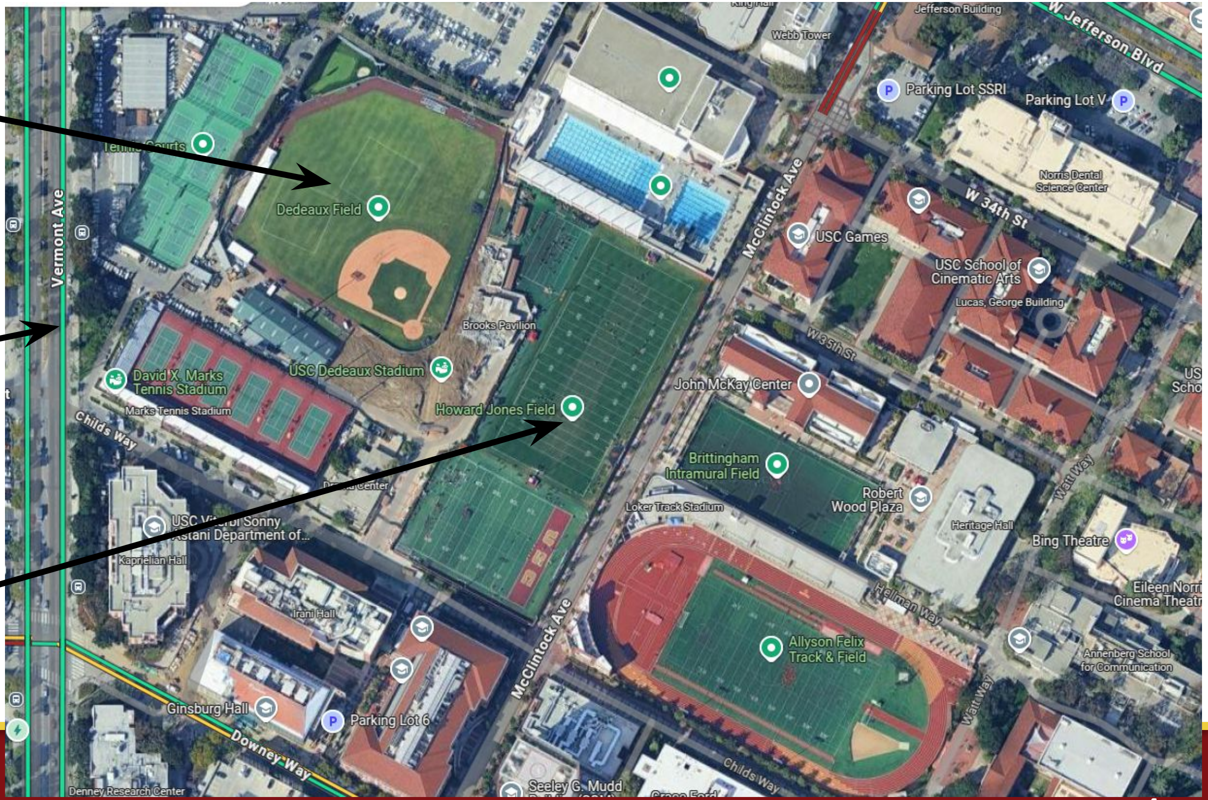
Computer Graphics

Many Applications

polygon

line

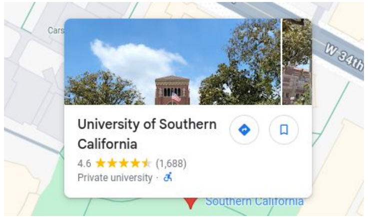
point



Fundamental Operations

- In computational geometry, the most primitive object is a point.

A = (0, 1)

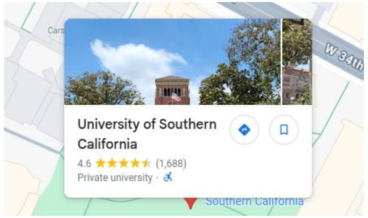


B = (34.022, -118.285)

Fundamental Operations

- In computational geometry, the most primitive object is a point.
- Common Operations: *addition, subtraction*

A = (0, 1)



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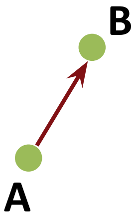
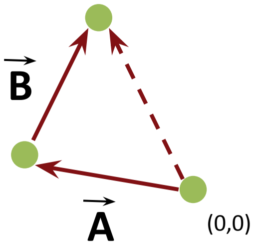


$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

subtraction

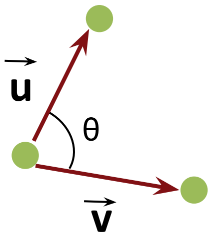


$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

addition

Fundamental Operations

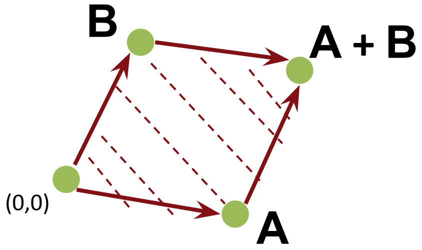
- In computational geometry, the most primitive object is a point.
- Common Operations: *addition, subtraction, dot product, cross product*



$$(x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1y_2$$

$$u \cdot v = \|u\|\|v\| \cos(\theta)$$

dot product



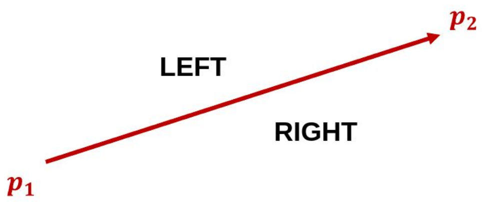
$$(x_1, y_1) \times (x_2, y_2) := \det\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right) = x_1y_2 - y_1x_2$$

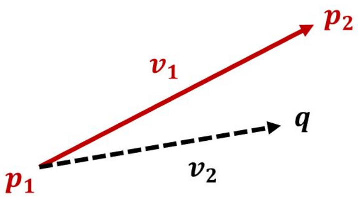
$$u \times v = \|u\|\|v\| \sin(\theta)$$

cross product

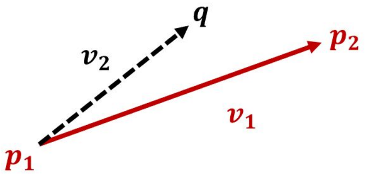
Line Side Test

- Decide whether a point *q* is on the left or right of a line segment





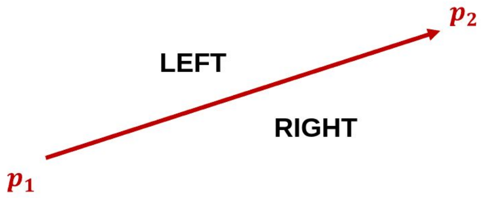
q is *right* of v_1

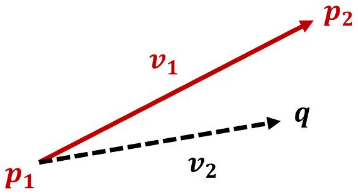


q is *left* of v_1

Line Side Test

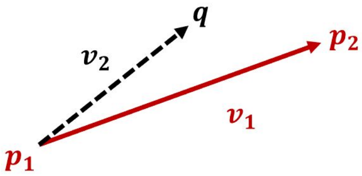
- Decide whether a point *q* is on the left or right of a line segment
  - Construct vectors: $v_1 = p_2 - p_1$ and $v_2 = q - p_1$
  - Compute the cross product v_1 and v_2
  - Compare value to 0





$v_1 \times v_2 < 0$

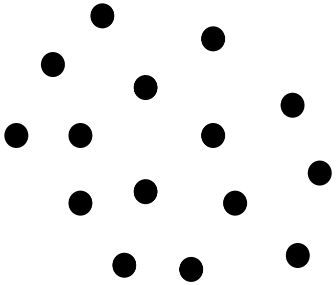
q is *right* of v_1



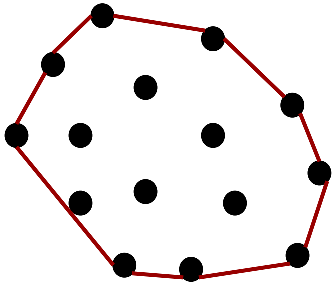
$v_1 \times v_2 > 0$

q is *left* of v_1

Convexity

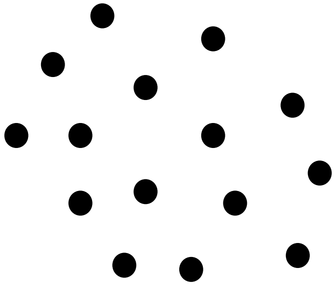


A set of points **P** in a Euclidean space



Convex Hull

Convexity



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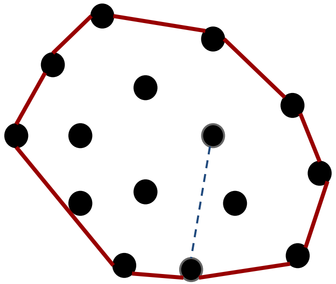
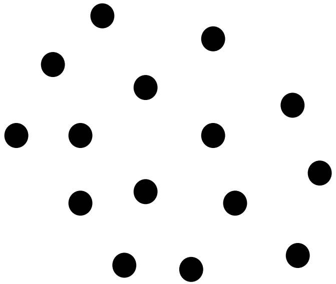


Convex Hull

A point set $P \subseteq R^d$ is convex if it is closed under convex combinations.

Convexity



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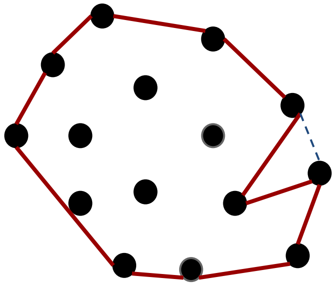
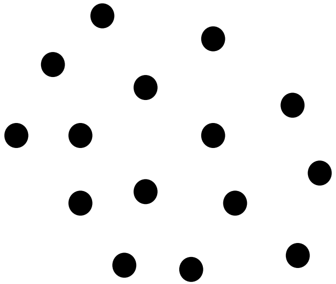


Concave

Convexity



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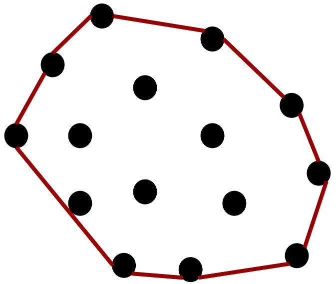
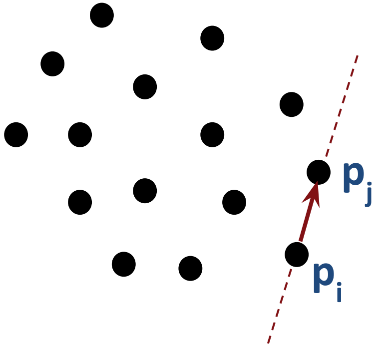


Convex Hull

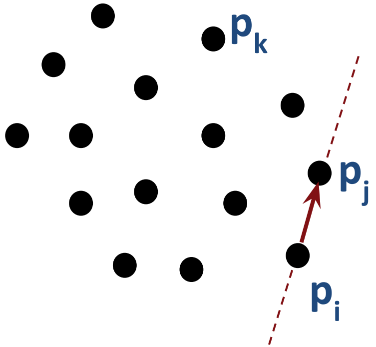
Definition: The convex hull of a set of points **P** is the boundary of the convex closure of **P** . That is, it is the *smallest convex polygon* that contains *all* of the points in **P** , either on its boundary or interior.

Algorithms for 2D Convex Hull



Claim: A directed segment between a pair of points p_i, p_j is <u>on</u> the convex hull if and only if all other points are to the left of the ray through p_i and p_j.

Algorithms for 2D Convex Hull



Brute Force Algorithm

1. Try every pair of points p_i, p_j
2. Perform Line Side Test on every other point p_k
3. If every p_k is on the left:
   a. Add $p_i \rightarrow p_j$ to the hull
4. Sort the final set of edges into *counterclockwise* order

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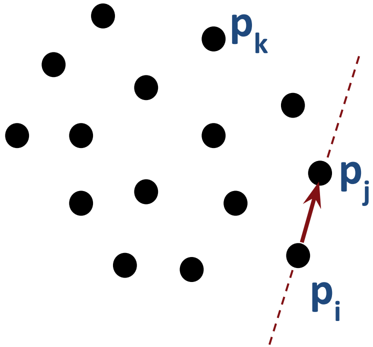
Algorithms for 2D Convex Hull



Brute Force Algorithm ⟹ **Complexity?**

1. Try every pair of points $\mathbf{p_i}$, $\mathbf{p_j}$
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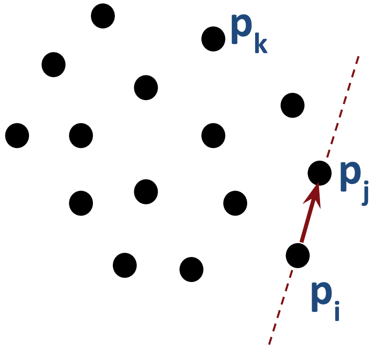
Algorithms for 2D Convex Hull



Brute Force Algorithm ⟹ $O(N^3)$

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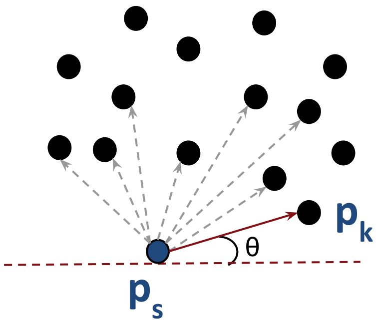
Algorithms for 2D Convex Hull

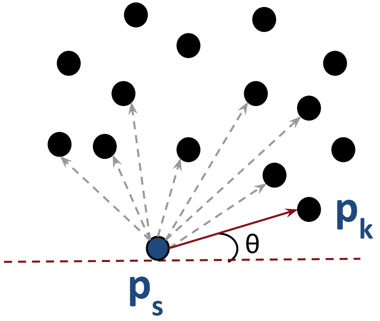


A little bit faster Algorithm

1. Take the point with the lowest y-coordinate $\mathbf{p}_s$
2. Measure the angle from $\mathbf{p}_s$ to all the other points $\mathbf{p}_k$
3. Select the point with the *smallest angle*
   a. Add $\mathbf{p}_s \rightarrow \mathbf{p}_k$ to the hull
4. Find the point $\mathbf{p}_u$ that has the smallest angle with respect to $(\mathbf{p}_s, \mathbf{p}_k)$
5. Continue until all points are exhausted

Claim: The lowest y-coordinate point $\mathbf{p}_s$ is *always* in the convex hull.

Algorithms for 2D Convex Hull



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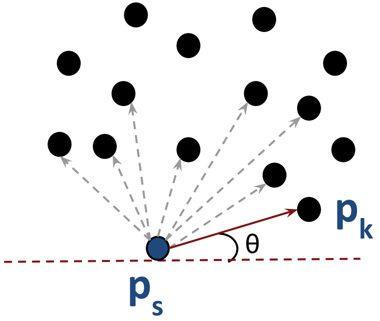
Algorithms for 2D Convex Hull



A little bit faster Algorithm ⟹ **$O(N^2)$**

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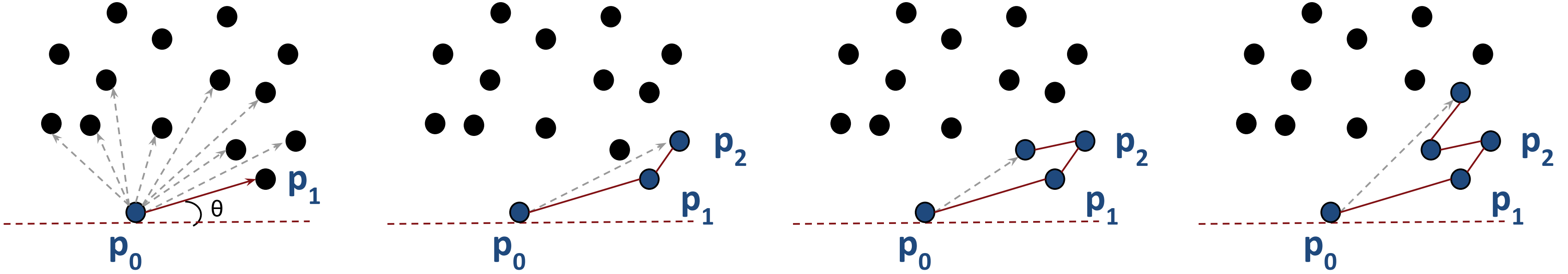
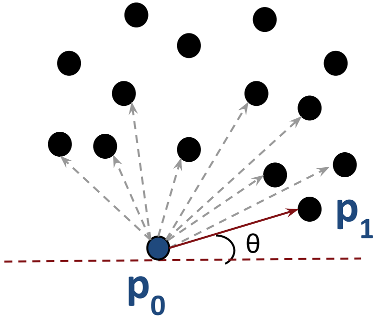
Algorithms for 2D Convex Hull



we are making a **right** turn!

Claim: The lowest y-coordinate point $\mathbf{p_s}$ is *always* in the convex hull.

Algorithms for 2D Convex Hull



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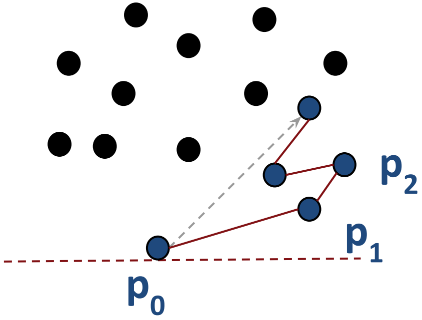

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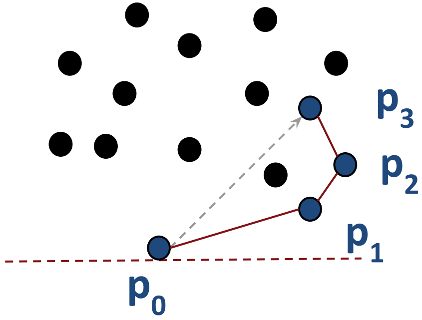
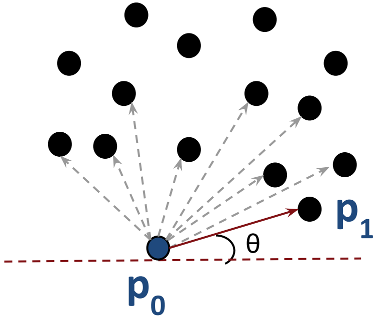


Remove the last added point from Hull

Claim: The lowest y-coordinate point $\mathbf{p_s}$ is *always* in the convex hull.

Algorithms for 2D Convex Hull

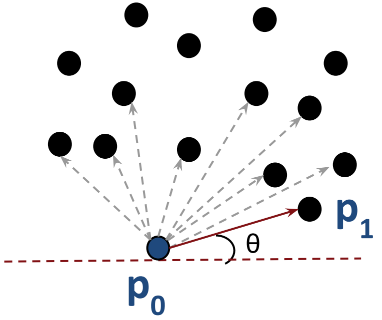


Graham Scan Algorithm

1. Find lowest y-coordinate point $\mathbf{p_0}$
2. Sort the points counterclockwise by *their angle with* $\boldsymbol{p_0}$
3. Hull = [$\mathbf{p_0}$, $\mathbf{p_1}$]
4. For each point $\mathbf{p_i}$
   a. If LineSideTest(Hull, $\mathbf{p_i}$) is RIGHT
      i. H.pop() // remove last element
   b. H.add($\mathbf{p_i}$)

Claim: The lowest y-coordinate point $\mathbf{p_s}$ is *always* in the convex hull.

Algorithms for 2D Convex Hull



Graham Scan Algorithm ⟹ **Complexity?**

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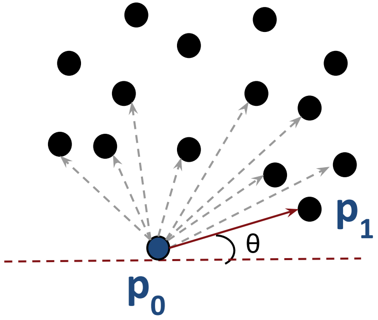
Algorithms for 2D Convex Hull



Graham Scan Algorithm ⟹ **O(NlogN)**

1. Find lowest y-coordinate point $\mathbf{p_0}$
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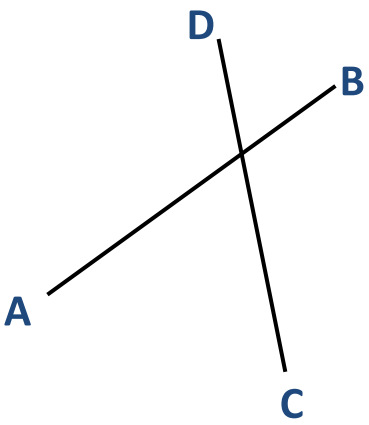
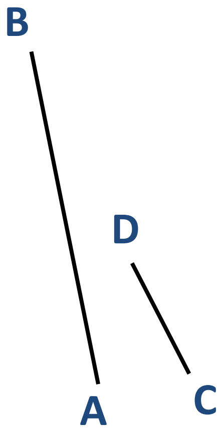
Claim: The lowest y-coordinate point $\mathbf{p_s}$ is *always* in the convex hull.

Intersections



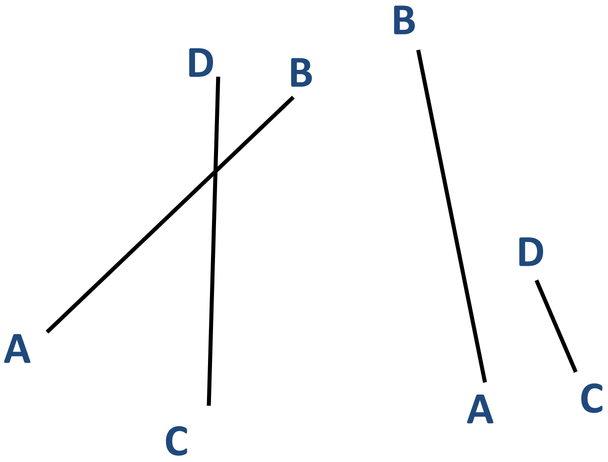
Intersecting pair



Non Intersecting pair

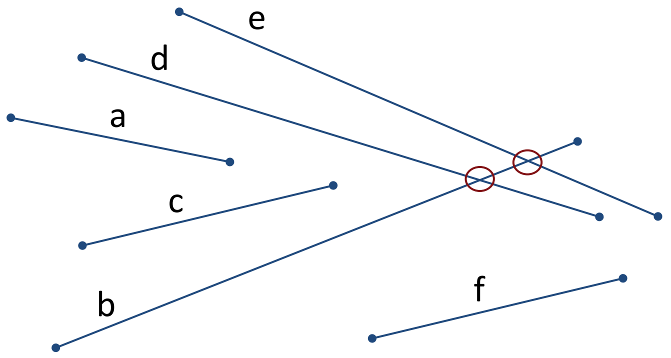
When do segments **AB** and **CD** *intersect*?

Intersections



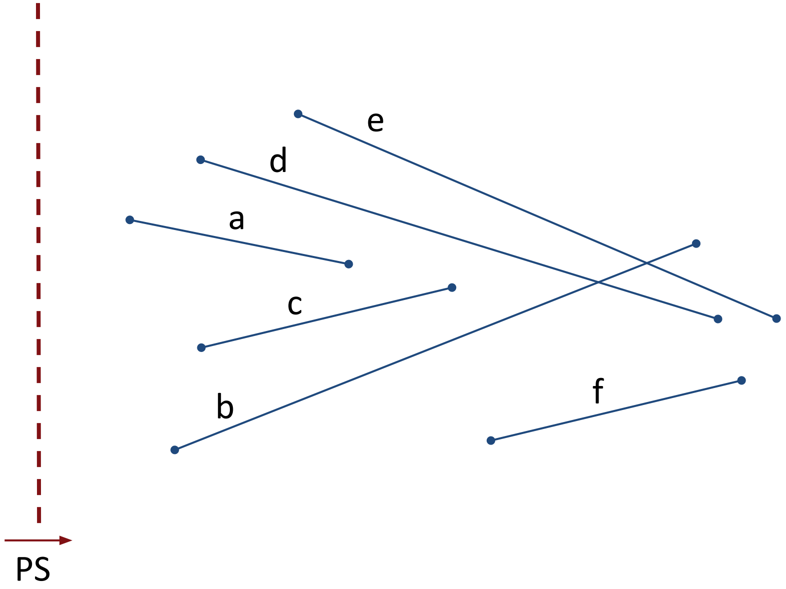
- When do segments **AB** and **CD** *intersect*?
  - We can take every point in one segment and test if it exists in the other.
  - We can check if A and B are on opposite sides of **CD** segment.
    - *eg.* LineSideTest(CD, A) = RIGHT and LineSideTest(CD, B) is LEFT

Intersections

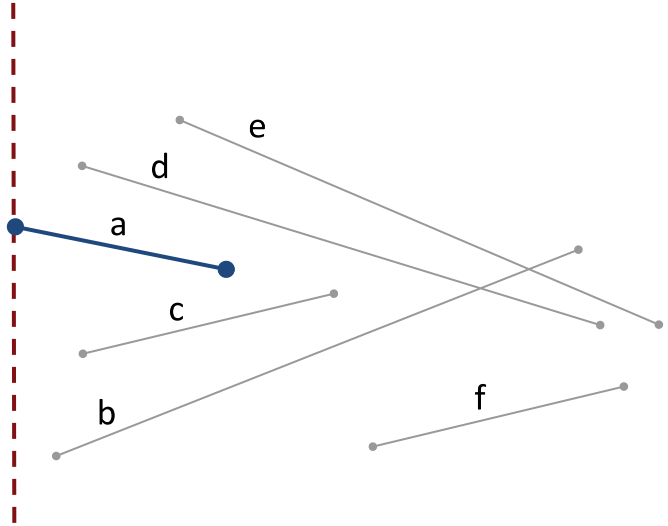


- What if we have **N** segments and want to detect **k** intersections?
  - We can perform the same checks for <u>all</u> possible pairs of segments.
  - **Complexity:** $O(N^2)$

Intersections: Plane Sweep Algorithm

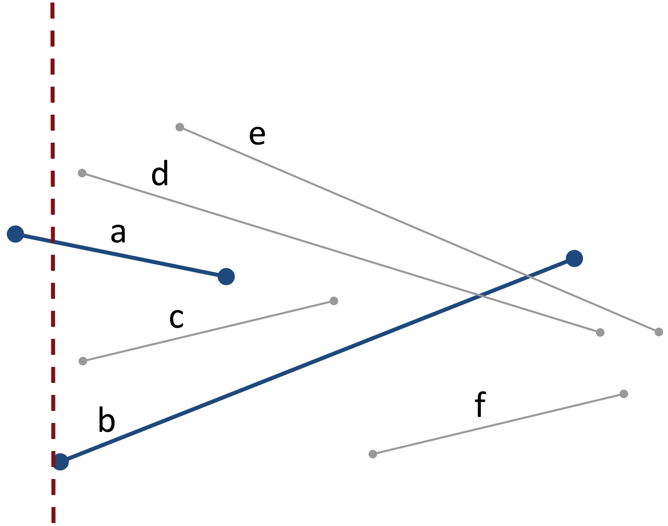


Intersections: Plane Sweep Algorithm



Currently exploring: a

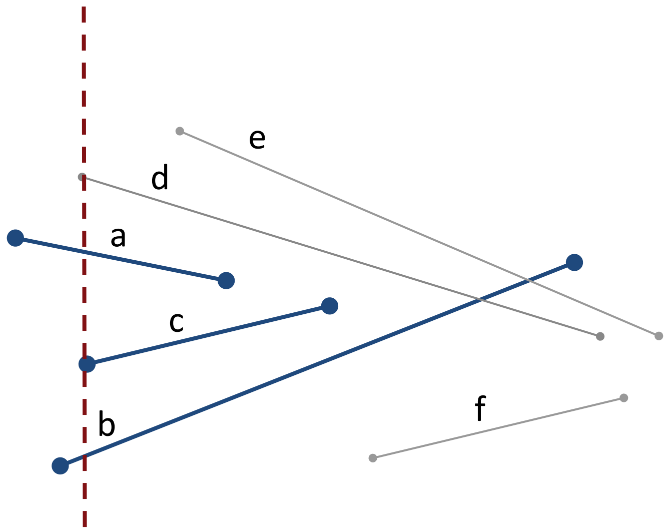
Intersections: Plane Sweep Algorithm



Currently exploring: b, a

➔ b does not *intersect* with a

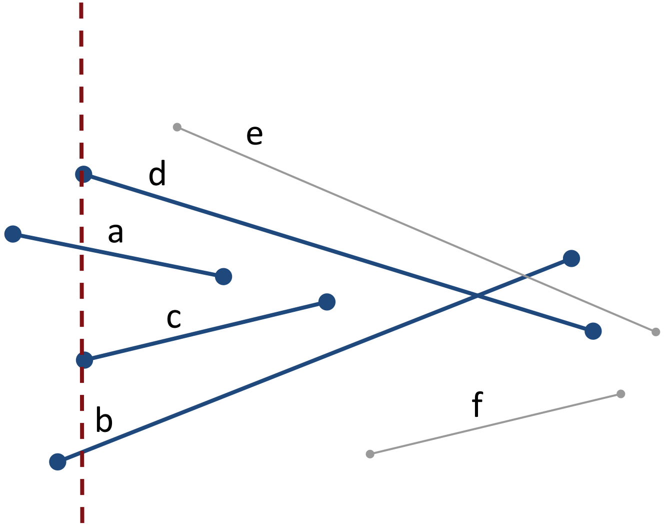
Intersections: Plane Sweep Algorithm



Currently exploring: b, c, a

➔ c does not *intersect* with b or a

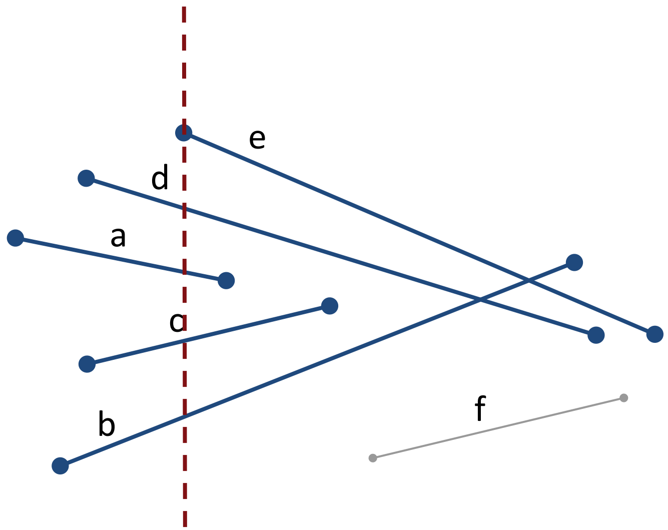
Intersections: Plane Sweep Algorithm



Currently exploring: b, c, a, d

➔ d does not *intersect* with a

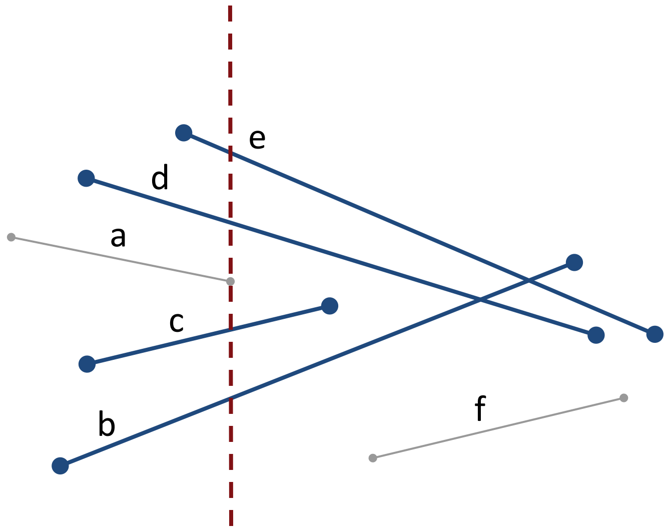
Intersections: Plane Sweep Algorithm



Currently exploring: b, c, a, d, e

➔ e does not *intersect* with d

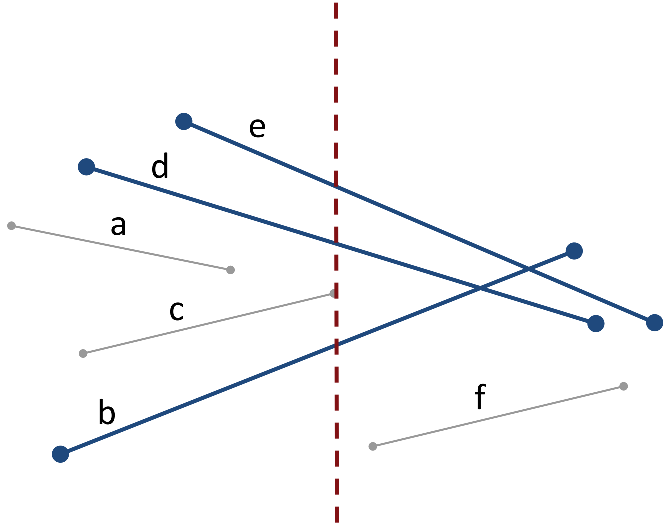
Intersections: Plane Sweep Algorithm



Currently exploring: b, c, ~~a~~, d, e

➔ c does not *intersect* with d !

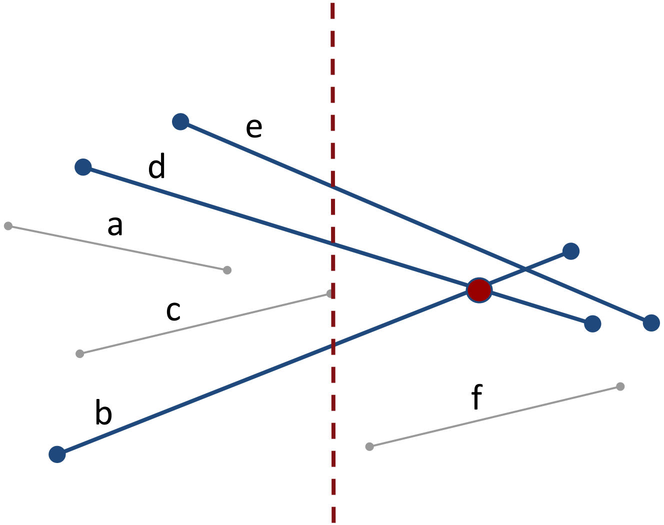
Intersections: Plane Sweep Algorithm



Currently exploring: b, ~~c~~, d, e

➔ b *intersects* with d !

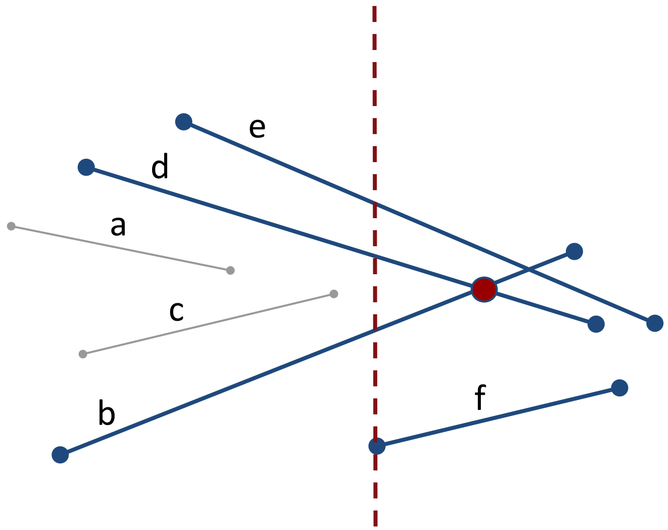
Intersections: Plane Sweep Algorithm



Currently exploring: b, d, e

➔ b *intersects* with d !

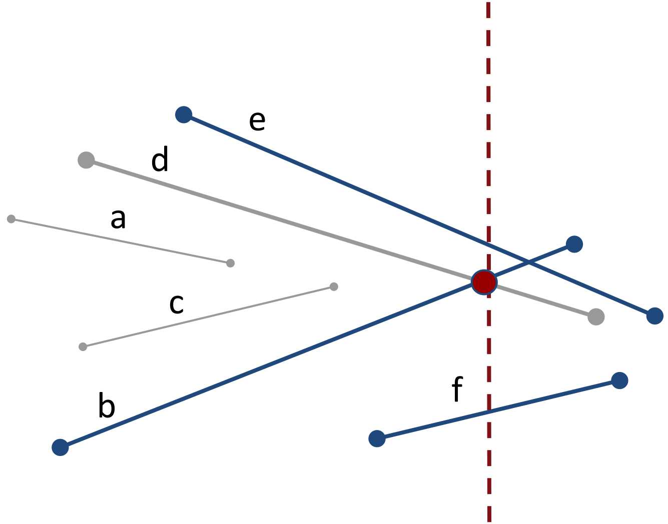
Intersections: Plane Sweep Algorithm



Currently exploring: f, b, d, e

➔ f does not *intersect* with b

Intersections: Plane Sweep Algorithm

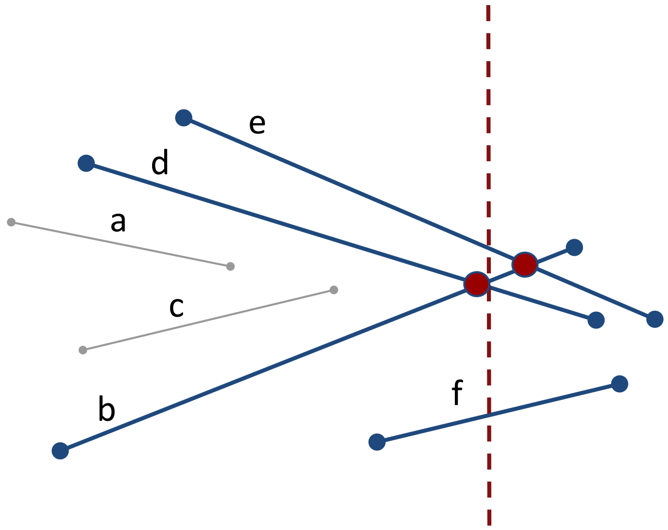


Currently exploring: f, b, ~~d~~, e

➔ e *intersects* with b !

Note: Treat intersection point as endpoint

Intersections: Plane Sweep Algorithm

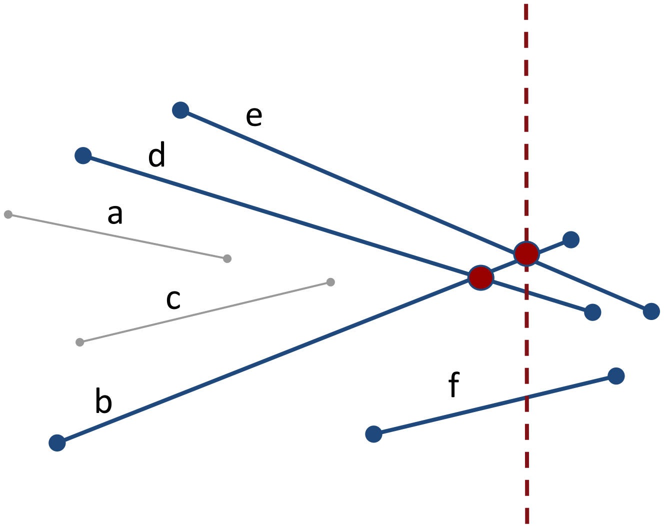


Currently exploring: f, b, d, e

→ e *intersects* with b !

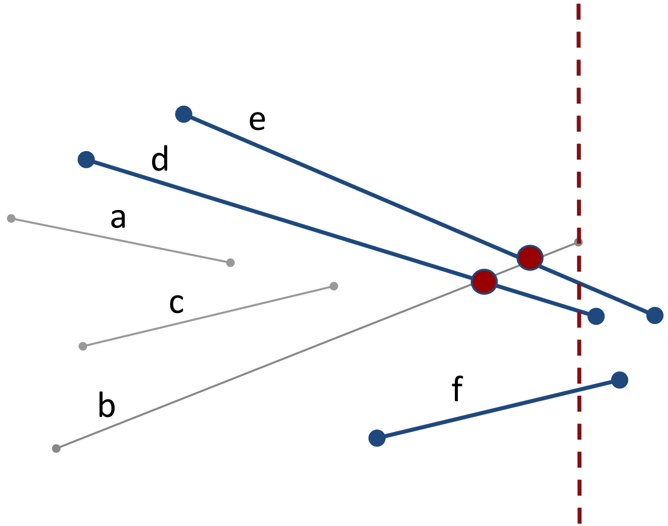
Note: Treat intersection point as endpoint

Intersections: Plane Sweep Algorithm



Currently exploring: f, b, d, e

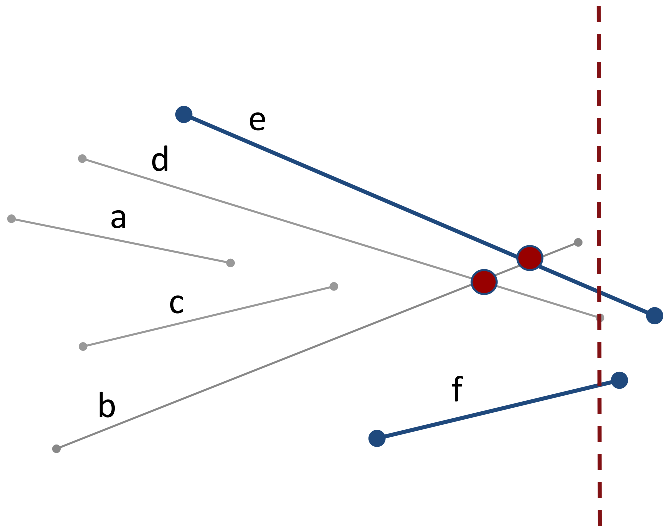
Intersections: Plane Sweep Algorithm



Currently exploring: f, d, e

➔ f does not *intersect* with d

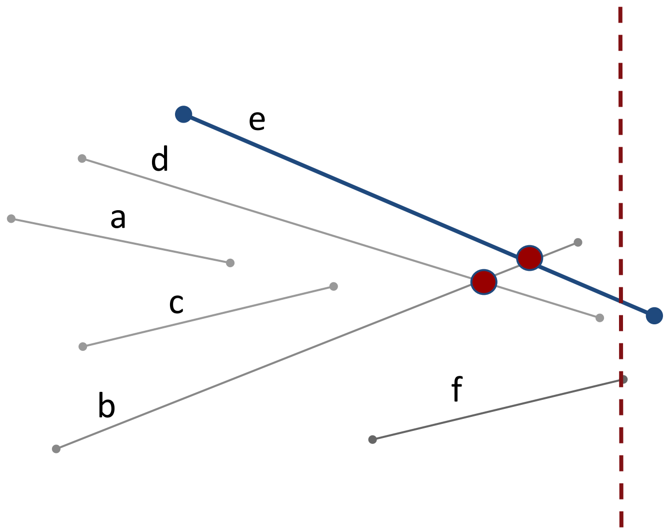
Intersections: Plane Sweep Algorithm



Currently exploring: f, e

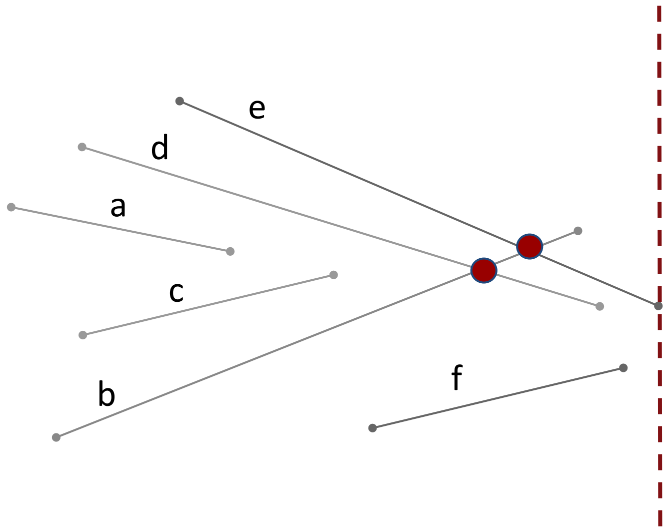
➔ f does not *intersect* with e

Intersections: Plane Sweep Algorithm



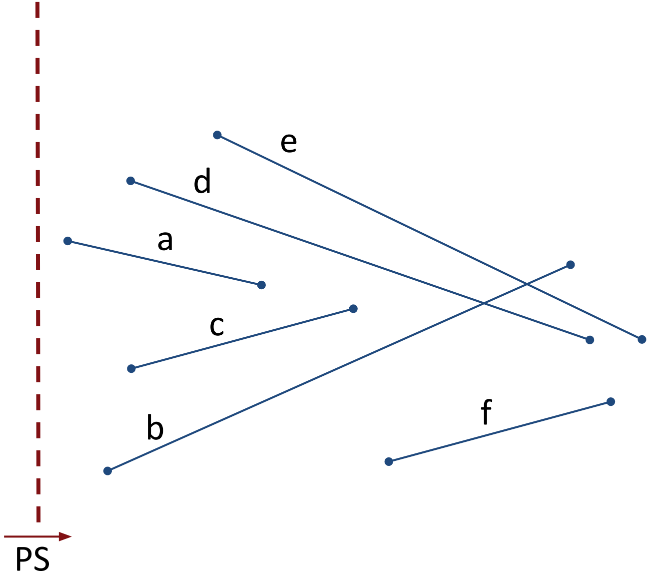
Currently exploring: e

Intersections: Plane Sweep Algorithm



Currently exploring:

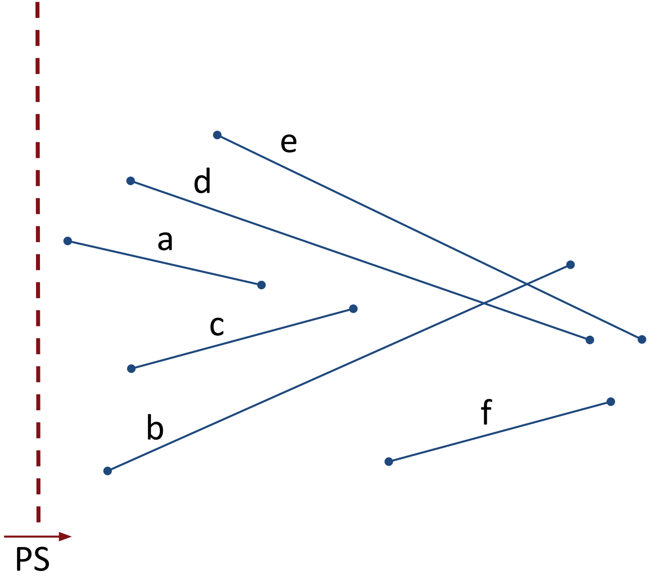
Intersections: Plane Sweep Algorithm



Plane Sweep Algorithm

1. Queue **EQ** = start and end of each segment *Si;* List **SL** = {}
2. For pi in **EQ**:
    a. If pi is *start* point:
        i. **SL**.add(pi); Intersects(Si, succ(Si)); Intersects(Si, predec(Si));
    b. If pi is *end* point:
        i. **SL**.delete(pi); Intersects(succ(Si), predec(Si));
    c. If *cross event* for Si, Sj:
        i. Remove Si from **SL**; Intersects(Sj, new neighbor);
        ii. Remove Sj from **SL**; Intersects(Si, new neighbor);

Intersections: Plane Sweep Algorithm



Plane Sweep Algorithm ⟹ **O(NlogN)**

1. Queue **EQ** = start and end of each segment *Si;* List **SL** = {}
2. For pi in **EQ**:
   a. If pi is *start* point:
      i. **SL**.add(pi); Intersects(Si, succ(Si)); Intersects(Si, predec(Si));
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   c. If *cross event* for Si, Sj:
      i. Remove Si from **SL**; Intersects(Sj, new neighbor);
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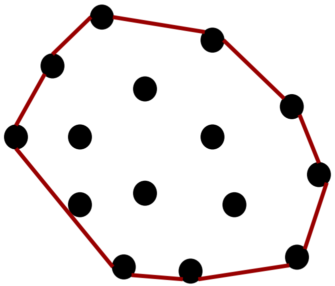
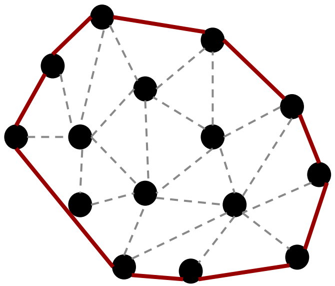
Triangulation



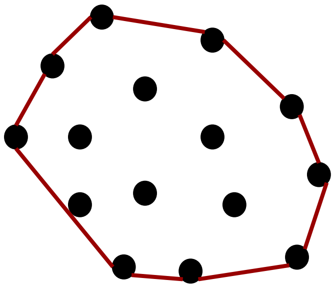
Convex Hull



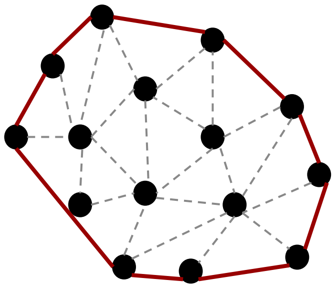
Triangulated Convex Hull

Definition: A **triangulation** is the process of subdividing a complex object (e.g. convex hull) into a disjoint collection of “simpler” objects (e.g. triangles).

Triangulation



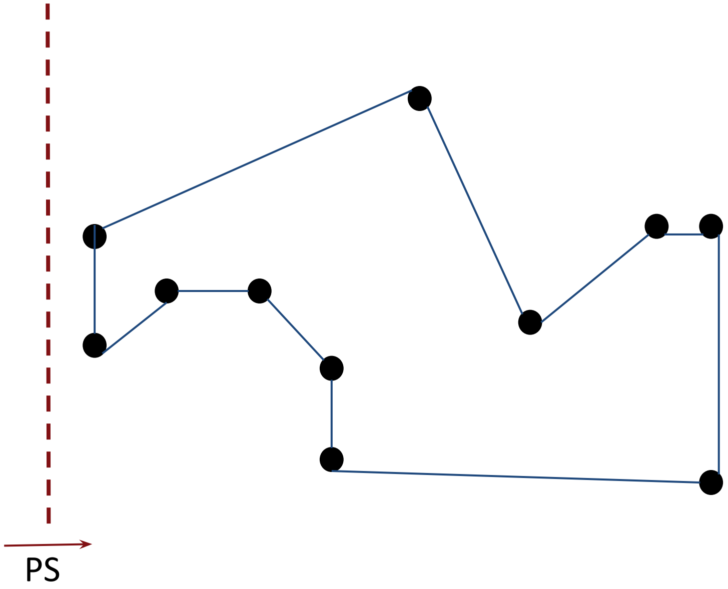
Convex Hull



Triangulated Convex Hull

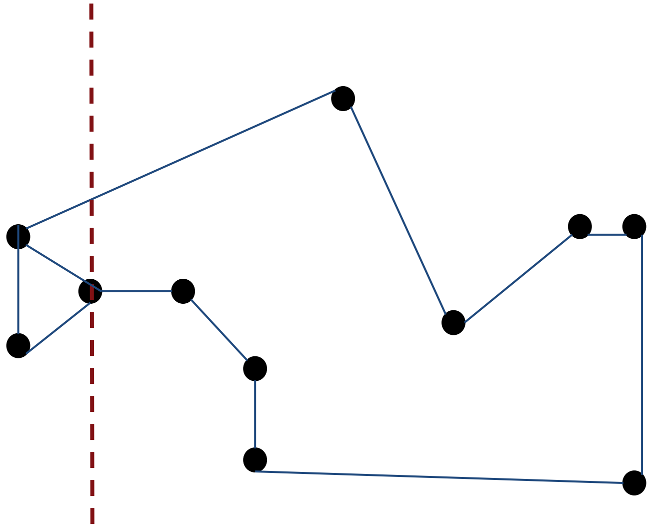
We can again use the **Plane Sweep Algorithm**!

Triangulation: Plane Sweep Algorithm

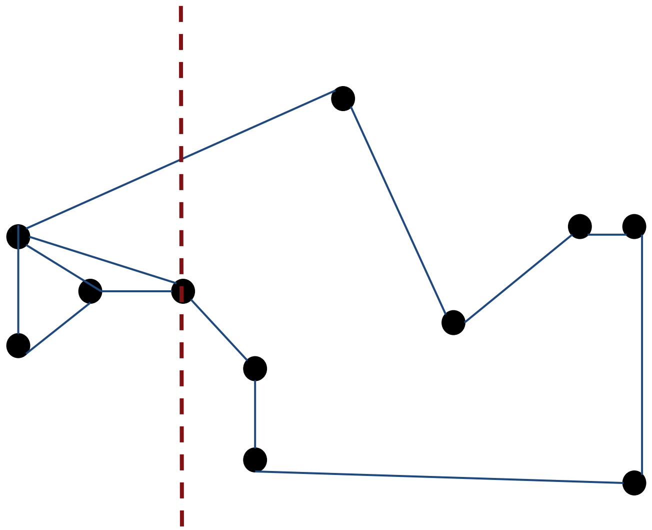


Intuition: Try to triangulate everything you can that is left from the sweep by *adding diagonals*.

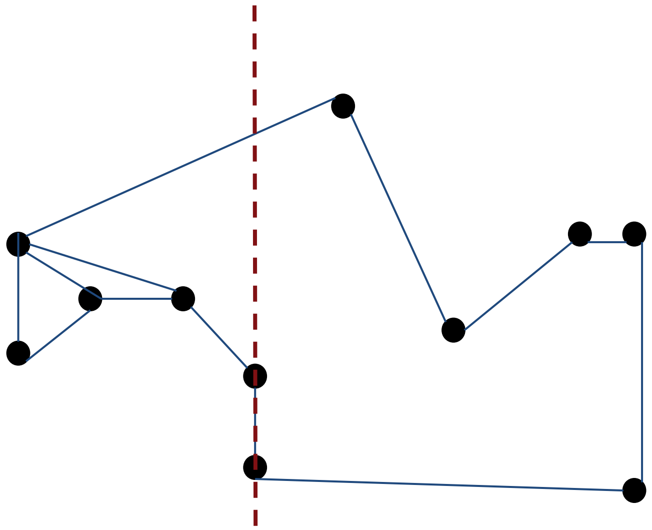
Triangulation: Plane Sweep Algorithm



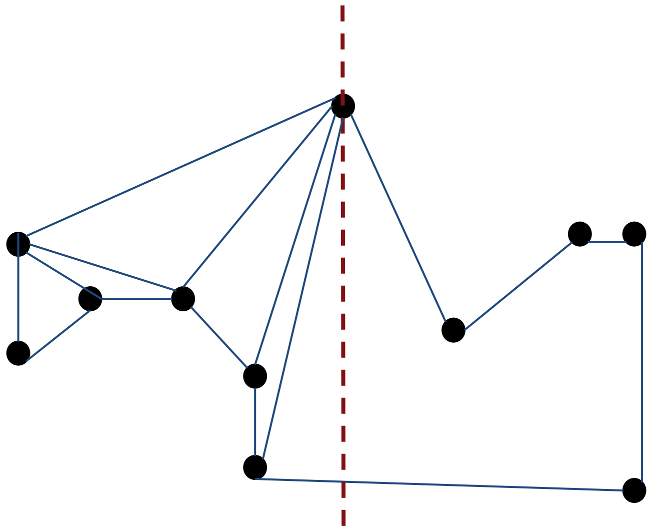
Triangulation: Plane Sweep Algorithm



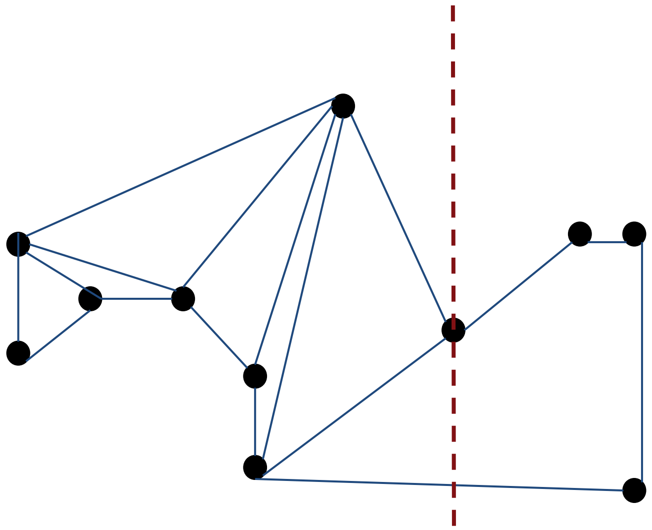
Triangulation: Plane Sweep Algorithm



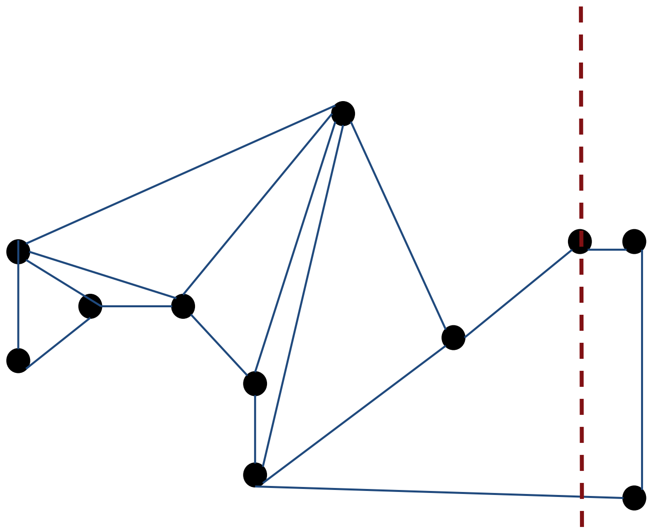
Triangulation: Plane Sweep Algorithm



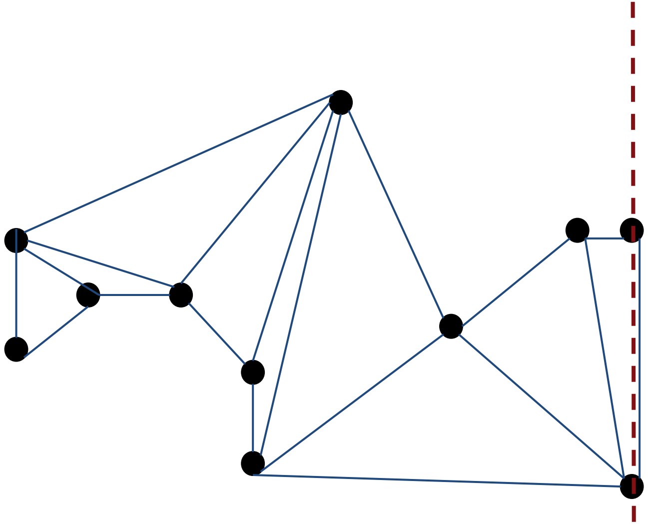
Triangulation: Plane Sweep Algorithm



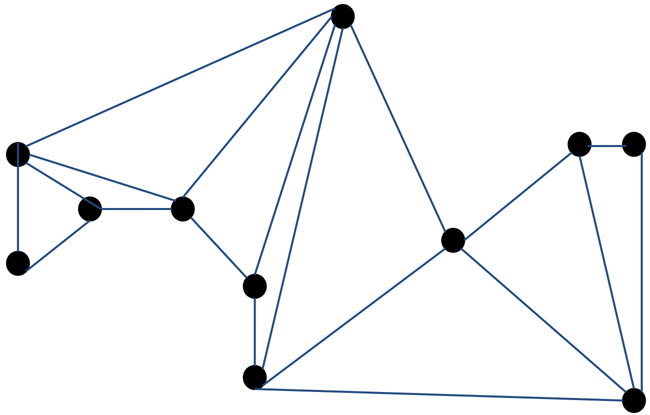
Triangulation: Plane Sweep Algorithm



Triangulation: Plane Sweep Algorithm

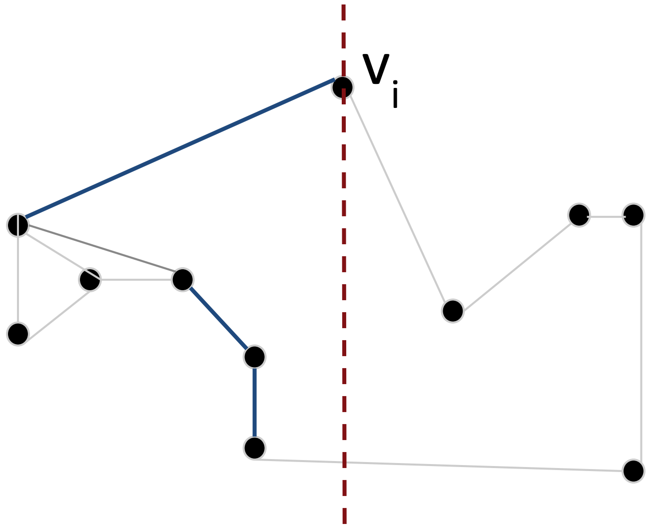


Triangulation: Plane Sweep Algorithm



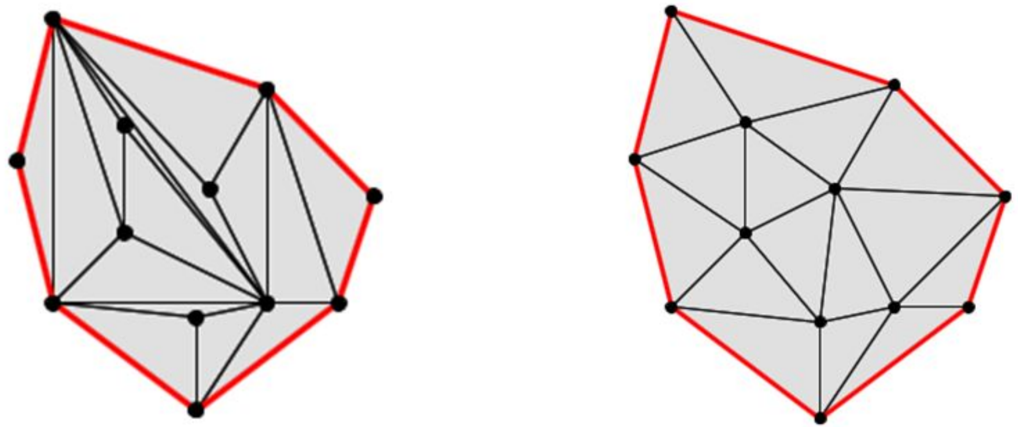
How can we determine if a region is ***un-triangulated***?

Triangulation: Plane Sweep Algorithm



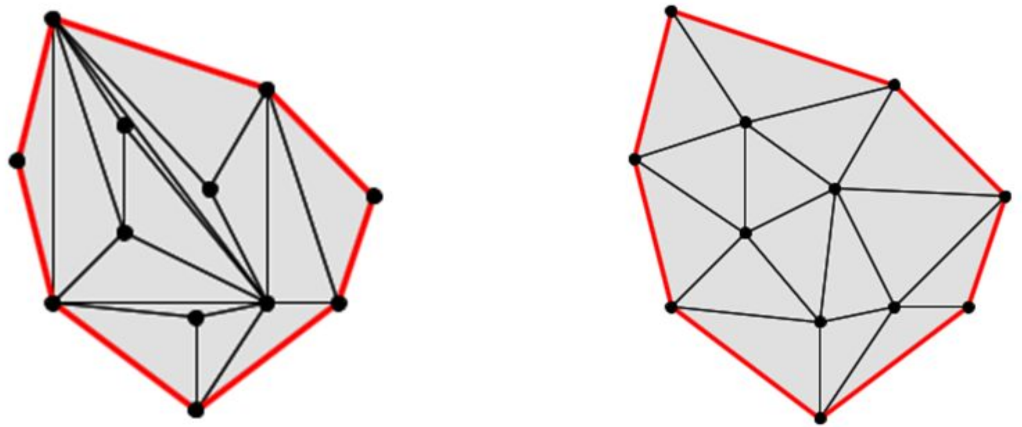
Lemma: For i≥2, after processing vertex v_i, **if** there are 2 x-monotone *chains*, a lower and an upper chain, and one has multiple edges, then this is an **untriangulated region**.

Triangulation: Delaunay Triangulation



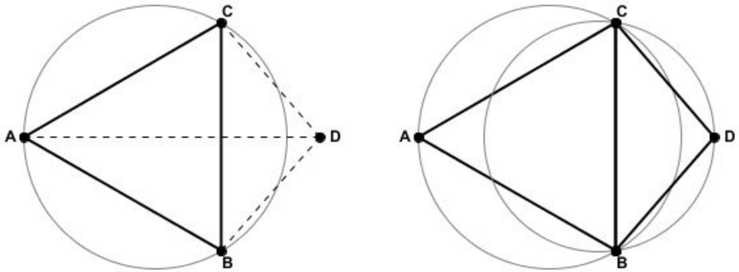
Triangulation might not be **optimal!**

Triangulation: Delaunay Triangulation

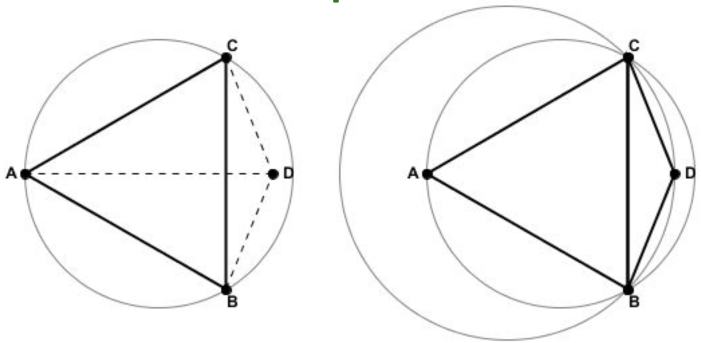


Triangulation might not be **optimal!**

Delaunay Triangulation

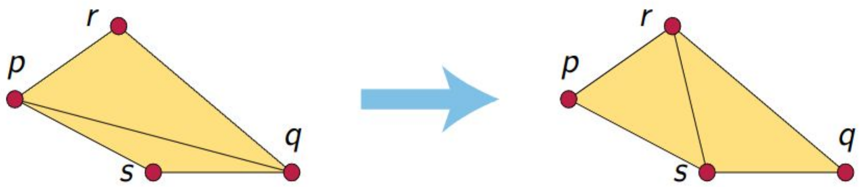


Accepted



Not Accepted

- ➔ Use any triangulation algorithm to triangulate a polygon, e.g. plane sweep triangulation.
- ➔ **Fix** bad triangulations afterwards.
  a. Triangle points should not be collinear.
  b. The circumcircle of each triangle not contain any other vertices from the mesh except for the three vertices that define the triangle.



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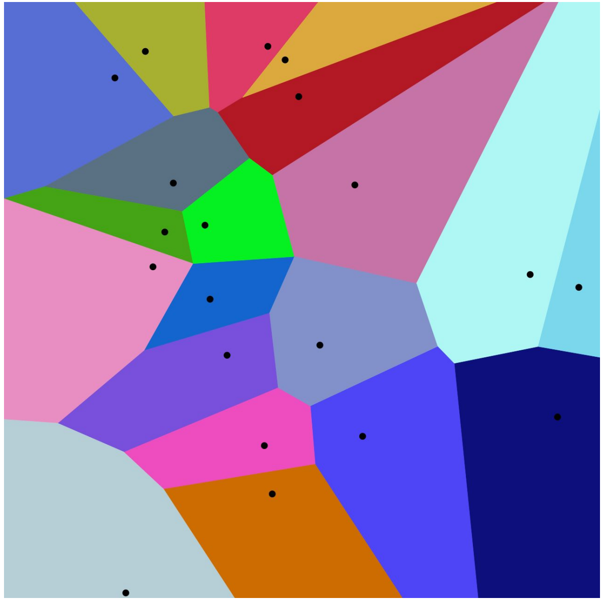
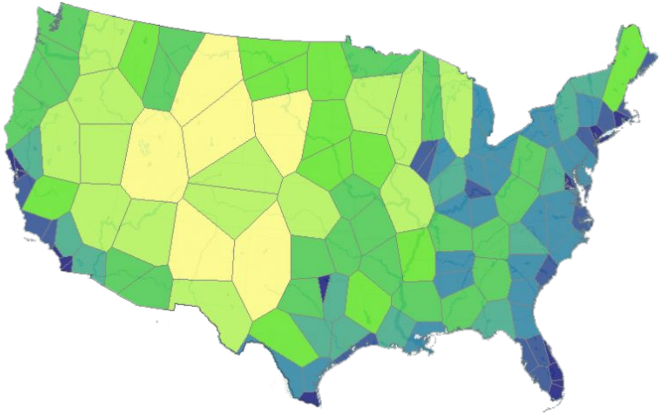
Voronoi Diagrams

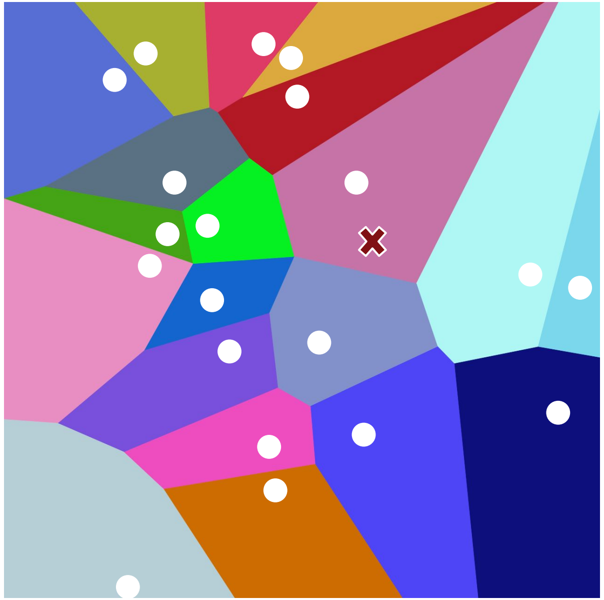




Voronoi Diagram of airports in the US

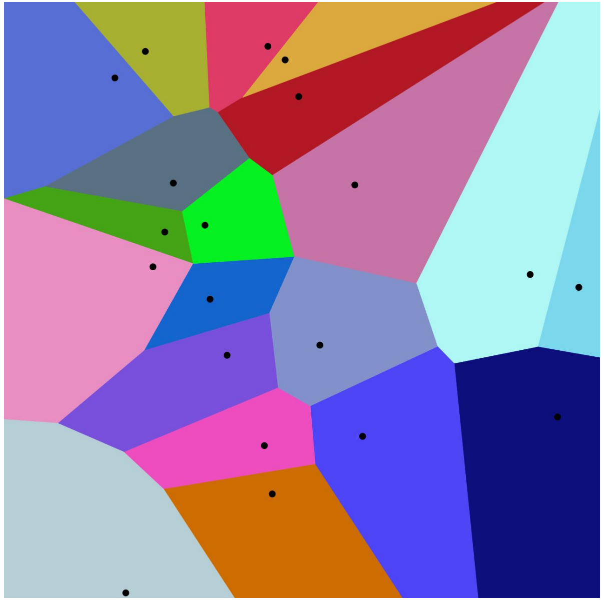
Definition: A **Voronoi diagram** is a partition of a plane into regions close to each a given set of objects.

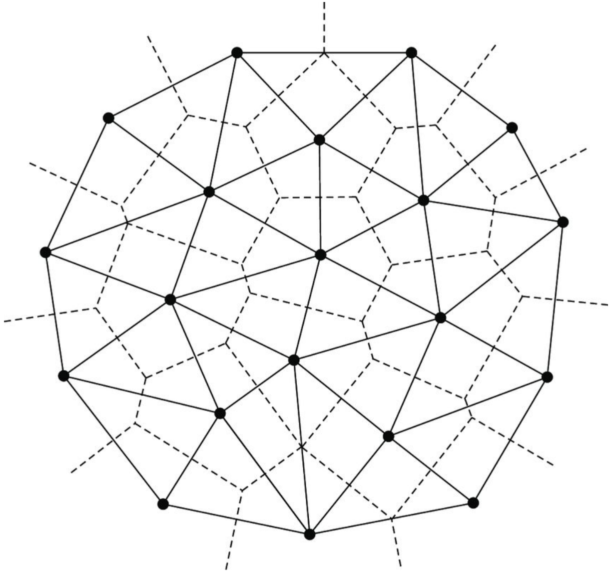
Voronoi Diagrams



Which is the closest POI to **X** ?

Voronoi Diagrams





Dual with Delaunay Triangulation

References

- CMU Fall 2022 Lectures on *Fundamentals of Computational Geometry* (https://www.cs.cmu.edu/~15451-f22/lectures/lec21-geometry.pdf)
- Duke Fall 2008 Lectures on Design and Analysis of Algorithms (https://courses.cs.duke.edu/fall08/cps230/Lectures/L-20.pdf)
- UCR CS133 Lectures on Computational Geometry (https://www.cs.ucr.edu/~eldawy/19SCS133/slides/CS133-05-Intersection.pdf)